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THE MATHEMATICS TEACHER

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Edited by William David Reeve

Interesting Detours in Algebraic Instruction

By S. F. Bibb

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Introduction. If this paper furnishes suggestions for further interesting search, then its purpose will have been fulfilled. Obviously a great amount of absorbing material could be properly mentioned. But included are only a few fascinating problems that have been passed on to the students when an opportunity afforded their demonstration.

The material is arranged under three headings: (1) Rational Numbers; (2) Irrational Numbers; (3) Functions and Their Zeros, in Particular The Quadratic Function.

Rational numbers. A theory of positive integers immediately suggests the question as to how their factors could be determined. And the notion of factors of positive integers projects the idea of prime numbers and perfect numbers into the discussion. For example the *prime numbers*

$$1, 2, 3, 5, 7, 11, 13, \cdots$$

consists of an unending sequence, some of which are given by

$$2^n + 1 \qquad (n \text{ a power of } 2),[1]$$

[1] L. E. Dickson, *Introduction to The Theory of Numbers*, p. 4.
J. W. A. Young, *Monographs on Modern Mathematics*, p. 308.
R. D. Carmichael, *The Theory of Numbers*, p. 28.

But $2^{32}+1 = 641 \times 6700417$.

And $2^{11}-1 = 23 \cdot 89$

and some are given by

$$2^p - 1 \qquad (p \text{ itself a prime}).^1$$

Surely this sequence of primes affords a natural and easy approach to the concept of an endless and ordered array of numbers. The student of his own volition will substitute values for n and p in order to compute some of the elements of this sequence, and thus the important idea of substitution may be introduced.

Similarly the *perfect numbers*

$$6, 28, 496, 8128, \cdots$$

consists of an endless sequence which may be determined by

$$2^{p-1}(2^p - 1) \qquad (\text{if } 2^p - 1 \text{ is a prime}).^2$$

Furthermore every perfect number is of this type, called Euclid's type.[2] This formula also will stimulate interest in substitution. Again, almost unaware of any effort on his part the student will get some definite mathematical relation existing between the prime numbers and the perfect numbers, since 2^p-1 is restricted to prime numbers.

All rational fractions, p/q where p and q are integers, are expressible not only as a repeating decimal but also they may be developed into a simple continued fraction with a definite number of partial quotients.[3] Thus, $1/5 = 0.1999$

$$1/9 = 0.111 \cdots = 1/9, \text{ one partial quotient;}$$

$$12/99 = 0.121212 \cdots = \frac{1}{8 + \frac{1}{4}}, \text{ two partial quotients;}$$

$$123/999 = 0.123123123 \cdots = \cfrac{1}{8 + \cfrac{1}{8 + \frac{1}{5}}}, \text{ three partial quotients.}$$

The process of developing p/q into a simple continued fraction is simply that of finding the Greatest Common Divisor of p and q. As an illustration

[2] L. E. Dickson, *Introduction to The Theory of Numbers*, pp. 4 and 5.

[3] F. Klein, *Elementary Mathematics;* Translated by Hedrick and Noble, p. 42. *Chrystal's Algebra*, Part II, p. 424.

$$123/999 = \cfrac{1}{\cfrac{999}{123}}, \qquad \text{and} \qquad \begin{array}{l} 123)999(8 \\ \underline{984} \\ 15)123(8 \\ \underline{120} \\ 3)15(5 \\ \underline{15} \end{array}$$

Hence

$$123/999 = \cfrac{1}{8 + \cfrac{1}{8 + \frac{1}{5}}}\,.$$

Since the three examples above are such that the number of integers in p and q, the number of integers in the repeating decimal, and the number of partial quotients in the simple continued fraction are the same, they serve to excite the curiosity to discover what 1234/9999 would be.

Irrational numbers. In this section the algebraic and the transcendental irrational numbers are briefly discussed. For example

$$x_1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \quad \text{and} \quad -x_2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

are irrational algebraic numbers. Furthermore they are the roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

The partial quotients of x_1 and x_2 repeat indefinitely as indicated, or otherwise it is clear that they would be rational numbers. And the greater the number of partial quotients that are used the nearer does the rational fraction approach the irrational value. As the rational value is computed for each successive partial quotient, it will be alternately greater than and less than the irrational number expressed by the simple continued fraction.

In addition

$$\text{(a)} \qquad 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

and

(b)
$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cdots}}}}}}$$

are the positive roots of

(A) $x^2 - 3 = 0$ and (B) $3x^2 + 2x - 12 = 0$

respectively.

These examples illustrate the fact that every simple periodic continued fraction is equal to one of the positive roots of a quadratic equation with rational coefficients.

In order to express a positive root of a quadratic equation in the above forms, the positive root of example (B) which is $\dfrac{\sqrt{37}-1}{3}$ is here presented as illustrating a method.

$(\sqrt{37} - 1)/3 = 1 + (\sqrt{37} - 4)/3$, 1 is the largest integer obtainable.

$(\sqrt{37} - 4)/3 = 7/(\sqrt{37} + 4) = \dfrac{1}{(\sqrt{37}+4)/7}$.

Similarly

$(\sqrt{37} + 4)/9 = 1 + (\sqrt{37} - 3)/7 = 1 + 4/(\sqrt{37} + 3)$, and

$(\sqrt{37} + 3)/4 = 2 + (\sqrt{37} - 5)/4 = 2 + 3/(\sqrt{37} + 5)$, and

$(\sqrt{37} + 5)/3 = 3 + (\sqrt{37} - 4)/3 = 3 + \left(\dfrac{\sqrt{37}-1}{3} - 1\right)$.

Then

$$\frac{\sqrt{37}-1}{3} = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\left(\cfrac{\sqrt{37}-1}{3}-1\right)}}},$$

by setting $x = \dfrac{\sqrt{37} - 1}{3}$ one gets

$$3x^2 + 2x - 12 = 0.$$

Not all continued fractions are periodic, and since π cannot be so expressed it is given here as an example.

$$\pi = 3.14159265 \cdots = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + 1/292 \cdots}}}.$$

If $3+1/7$ and $3+\dfrac{1}{7 + 1/15}$, or the first and second partial quotients, are used the approximations

$$\pi = 22/7 = 3.14285 \text{ and } \pi = 333/106 = 3.141509$$ 355/113

are obtained.[4]

After demonstrating the existence of algebraic irrational numbers, a question as to whether there exists other irrational numbers is surely not lacking in interest. Students will listen with absorbing intent to an explanation of the transcendency of e and π; even though the bravest teacher would not venture a proof that they will not satisfy an algebraic equation with rational coefficients. That is, no such equations as

$$a_0 + a_1e + a_2e^2 + a_3e^3 + \cdots a_ne^n = 0, \text{ or}$$
$$b_0 + b_1\pi + b_2\pi^2 + b_3\pi^3 + \cdots b_n\pi^n = 0$$

exist where $a_0 \neq 0$, $b_0 \neq 0$, and all the a's and b's are rational numbers.

Furthermore, the fact that the proof of the transcendency of e (given in 1873 by Hermite)[5] was given before such proof was demonstrated for π (given in 1882 by Lindemann)[5] leads naturally to the mentioning of the remarkable relation

$$e^{i\pi} = -1.$$

Since the proof of this simple dependency follows so easily from

[4] F. Klein, *Elementary Mathematics*, p. 43. *Chrystal's Algebra*, Part II, p. 433.
[5] F. Klein, *Elementary Mathematics*, p. 238.

Euler's Relation for trigonometric functions, one could not refrain from giving it. Thus

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

and by addition

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which for $\theta = \pi$ gives $e^{i\pi} = -1$.[5]

Why mention only e and π? It is known that the totality of transcendental numbers form a field of numbers with a far

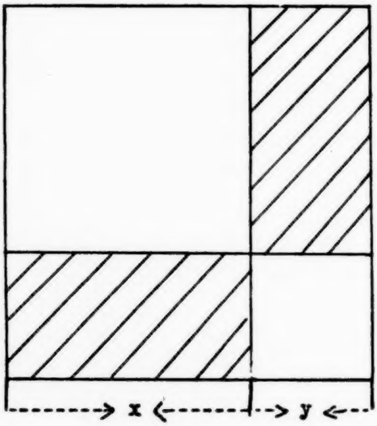

Fig. 1

greater number of elements than the elements of an algebraic field. Yet, the task of actually finding those transcendental numbers individually has baffled the ingenuity of mathematicians all these centuries. As late as 1900 Hilbert renewed interest in the problem, "*to prove or disprove that* $2^{\sqrt{2}}$ *is transcendental.*" Who knows but that among the listeners of a particular discussion there may be another unaroused Kusmin who in 1930 proved a whole infinity of numbers, one of which was $2^{\sqrt{2}}$, to be transcendental.[6]

Functions and their zeros. The Binomial theorem including the Pascal Triangle; the integral solutions of $x^n + y^n = z^n$; and the quadratic function in one variable are discussed in this section.

[6] E. T. Bell, *The Queen of The Sciences*, p. 100.

In books on algebra the expansions

$$(x + y)^2 = x^2 + 2xy + y^2$$

and

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

are given. On the other hand simple geometric equivalents of these two expansions are seldom given. *Thus the area of a square whose*

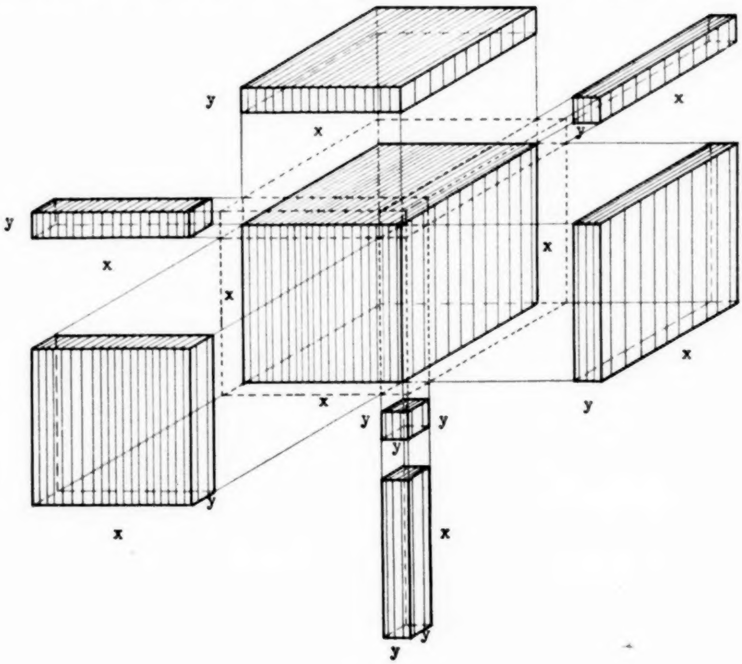

Fig. 2

side is $(x+y)$ *is easily seen to be the first of the above expansions.* See Fig. 1.

Similarly the volume of a cube whose edge is $(x+y)$ *is the picture of the second expansion given above.* See Fig. 2.

One need not stop with these two examples. The area of a square whose side is $(x+y+z)$ is forcibly pictured in Fig. 3.

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.$$

These examples are only suggestive and so long as $(x+y$

$+z+\cdots$) is of the second or third degree these geometrical proofs could be extended indefinitely.[7]

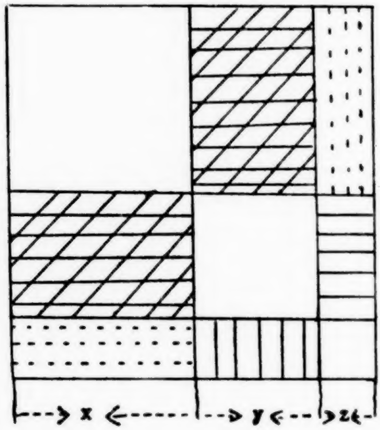

Fig. 3

The expansion,

$$(x+y)^n = x^n + {}_nC_1yx^{n-1} + {}_nC_2y^2x^{n-2} + \cdots + {}_nC_{n-1}y^{n-1}x + {}_nC_ny^n$$

is usually given. But the Arithmetical Triangle,

$(a+b)^0$	1						
$(a+b)^1$	1	1					
$(a+b)^2$	1	2	1				
$(a+b)^3$	1	3	3	1			
$(a+b)^4$	1	4	6	4	1		
$(a+b)^5$	1	5	10	10	5	1	
$(a+b)^6$	1	6	15	20	15	6	1
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
$(a+b)^n$	1	${}_nC_1$	${}_nC_2$	${}_nC_3$	$\cdots$	${}_nC_{n-1}$	${}_nC_n$

which is known as The Pascal Triangle does not often appear.

It is well known that any element in a diagonal of this triangle may be got by adding together two other elements, the one just above it and the one immediately to the left of this one. That is

$${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}.$$

[7] T. Percy Nunn, *The Teaching of Algebra*, Chapter XX.

Thus,

$${}_nC_r = n(n-1)(n-2)(n-3)(\cdots)(n-r+1)/r!$$

and

$${}_nC_{r+1} = n(n-1)(n-2)(n-3)(\cdots)(n-r+1)(n-r)/(r+1)!.$$

By adding together these two combinations one gets

$$\frac{n(n-1)(n-2)(n-3)(\cdots)(n-r+1)}{r!}\left(1+\frac{n-r}{r+1}\right),$$

or

$$\frac{(n+1)(n)(n-1)(n-2)\cdots(n-r+1)}{(r+1)!} = {}_{n+1}C_{r+1}.$$

Also it was recently proposed "*to prove that the difference of the squares of the two consecutive numbers in the third diagonal of the Pascal Triangle*

$$1, 3, 6, 10, 15, 21, \cdots$$

is always a perfect cube."[8] The proof is easy. Thus, the difference of the squares of the $(i+1)$th and $(i+2)$th elements of the third diagonal is

$$(i+2)^2(i+3)^2/4 - (i+1)^2(i+2)^2/4 = (i+2)^3.$$

For the $(r+1)$th diagonal is

$${}_{n+r}C_r \qquad (r = 1, 2, 3, \cdots).$$

The $(r+1)$th diagonal just given, ${}_{n+r}C_r$, also determines the number of coefficients of the Hypersurface

$$f(x_0, x_1, x_2, x_3, \cdots, x_r) = 0,$$

where n is the degree of the surface and r is the number of variables.[9] For example the *quadric surface* in 3-space in homogeneous coordinates

$$f(x_0, x_1, x_2, x_3) = 0$$

will have ${}_{2+3}C_3$ coefficients, or 10. Similarly the *cubic surface* in 3-space will have ${}_{3+3}C_3$ coefficients, or 20.

[8] *The American Mathematical Monthly*, Feb. 1933, p. 110.

[9] Eugenio Bertini, *Geometria Proiettiva Degle Iperspazi*, p. 189.

All of the solutions of

(1) $$a^2 + b^2 = c^2$$

where a, b, and c are restricted to be integers are obtained if

(2) $a = q^2 - p^2,\ b = 2pq,\ c = q^2 + p^2$ (p and q integers).

This important relation for finding all *Pythagorean numbers* is not difficult to derive. Therefore, if

(3) $$a/c = x \text{ and } b/c = y$$

equation (1) becomes

(4) $$x^2 + y^2 = 1.$$

The graph of this equation is a circle with center (0, 0) and one solution is $x = -1$, $y = 0$. The equation of all lines passing through $(-1, 0)$ and slope m is

(5) $$y = m(x + 1).$$

By solving (4) and (5) simultaneously for x the result is

(6) $$x^2 + 2m^2x/(1 + m^2) + (m^2 - 1)/(m^2 + 1) = 0.$$

One of the solutions of (6) is $x_1 = -1$. Let x_2 be the other solution. Then

(7) $$x_1 + x_2 = -2m^2/(1 + m^2),$$

and by setting $x_1 = -1$

(8) $$x_2 = (1 - m^2)/(1 + m^2).$$

Suppose y_2 is the ordinate of the point $P_2(x_2, y_2)$. Fig. 4. Then from (5)

(9) $$y_2 = m(x_2 + 1) = 2m/(1 + m^2).$$

Therefore, in accordance with (8) and (9) if m is a rational number, so ~~is~~ are x_2 and y_2.

Also (8) *and* (9) *give all the rational points of intersections of the line L and the circle C, if m is rational.*

In order to reduce to integers, set $m = p/q$ then

$$a/c = x_2 = (q^2 - p^2)/(q^2 + p^2)$$
$$b/c = y_2 = 2pq/(q^2 + p^2).$$

From these expressions (2) is obvious.[10]

To summarize, all Pythagorean numbers are given by the equations

$$a = q^2 - p^2, \quad b = 2pq, \quad c = q^2 + p^2;$$

and one gets the totality of solutions which have no common divisor if p and q take all pairs of relatively prime integral values.

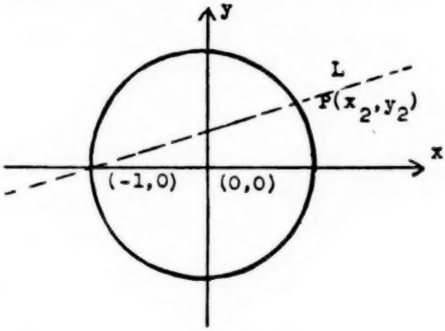

Fig. 4

One would hasten to investigate and hope to derive a similar result for

(10) $\quad a^3 + b^3 = c^3, \quad a^4 + b^4 = c^4, \cdots \quad a^n + b^n = c^n.$

From (3) the first two equations of (10) become

(11) $\quad x^3 + y^3 = 1$ and $x^4 + y^4 = 1,$

whose graphs are Fig. 5 and Fig. 6 respectively.

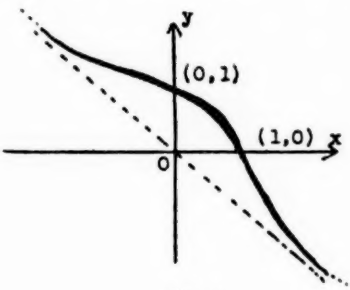

Fig. 5

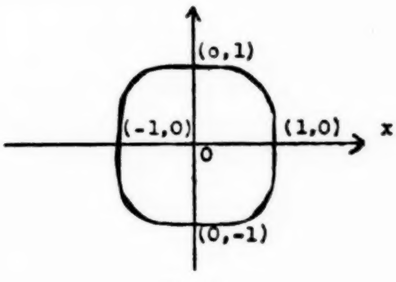

Fig. 6

[10] F. Klein, *Elementary Mathematics*, p. 44.

The unproved theorem of Fermat states that, unlike the circle in Fig. 4, the curves in Fig. 5 and Fig. 6 thread their way through the totality of points of the plane without passing through a single rational point other than those trivial ones indicated on the graphs. Furthermore, *Fermat's theorem states that in general* $a^n+b^n=c^n$ *has no integral solutions for integral values of* n *except when* $n=1$ *and* $n=2$.[11]

A discussion and the graph of a quadratic function in one variable

$$y = x^2 + 2px + q$$

affords a happy introduction to the very important notion of the relative maximum (minimum) value of a given function on a particular range. For example, let it be required to determine the shape of a rectangle so that the area is a maximum, provided its perimeter is $4p$ feet.

Students may guess that the required rectangle would be a square. Guesses are usually wrong; but this time why not show that the students guessed correctly? Thus, the area y (see Fig. 7) is

$$\begin{aligned} y &= x(2p - x) \\ &= 2px - x^2 \\ &= -(x^2 - 2px + p^2) + p^2. \end{aligned}$$

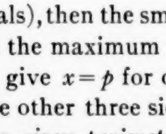

Fig. 7

This gives $(x-p)^2=p^2-y$.

Since $(x-p)^2$ is zero or positive (in field of reals), then the smallest value p^2-y could have would be zero, that is the maximum value for y would be p^2. And this value for y would give $x=p$ for one of the sides. Since the total length is to be $4p$, the other three sides of the rectangle must also equal p. *Hence with a given perimeter, the rectangle with a maximum bounded area is a square.*

The graph shows (see Fig. 8) that at the point (p, p^2) a tangent to the curve is parallel to the x-axis. Here arises an opportunity for explaining an interpretation of the first derivative of y with respect to x (dy/dx).

If a search for an introduction to the solution of a quadratic equation in one variable is worthy of note, surely then the graphic method is a fortunate find. To illustrate a method, if

[11] F. Klein, *Elementary Mathematics*, p. 47.

(12) $$x^2 - 2ax + b = 0$$

has a and b as real coefficients, and also if it has a real solution, these solutions may be constructed by the use of ruler and compasses; that is by the method of elementary geometry.

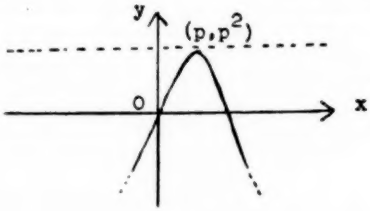

Fig. 8

One method of *constructing the roots of* (12) is as follows: Suppose a and b both positive. Choose one end of the diameter of a circle as the point $A(0, 1)$, the other end as $B(2a, b)$. *Where this circle crosses the x-axis, the determined abscissas, OC and OD, are the required roots.*[12] Proof,

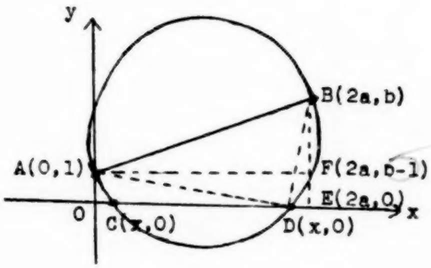

Fig. 9

(i) $(AD)^2 = 1 + x^2,$

(ii) $(DB)^2 = b^2 + (2a - x)^2,$

(iii) $(BA)^2 = (AF)^2 + (FB)^2 = (2a)^2 + (b - 1)^2$, also

(iv) $(BA)^2 = (AD)^2 + (DB)^2$ since arc ADB is a semi-circle.

By substituting (i), (ii), and (ii) in (iv)

$$(2a)^2 + (b - 1)^2 = 1 + x^2 + b^2 + (2a - x)^2,$$

or

[12] L. E. Dickson, *Elementary Theory of Equations*, p. 16. First Edition.

$$x^2 - 2ax + b = 0.$$

The same result would have been found if the point C instead of the point D had been used.

Hence the lengths OC and OD determine the roots of (12).

The better students will complete the solution if one or both of the roots are negative, instead of both positive as in the proof above given.[13]

CONCLUSION. At Armour Institute of Technology the librarians find it difficult to supply the demand for books in which the material of this paper is discussed.

Teachers everywhere question the final disposition of mathematics as a part of the curricula in the schools of the United States. But it is hoped that some such ideas as suggested herein may prove to be a timely stimuli; and that the students themselves guided by this or similar material will demand even more mathematics than is taught at present.

If the teachers would assume that their *task is to arouse a fascinating interest* in their subject, then the association between instructor and student would be *positively* successful.

[13] *The American Mathematical Monthly,* Feb. 1933, p. 81; a general reference.

Are We Teaching Arithmetic Effectively? A Summary of a Recent Study*

By Arthur E. Robinson

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Arithmetic was introduced into the elementary school in 1548. For 400 years it has held a place of importance in the elementary school curriculum second only to that of reading. Yet in spite of its venerable age, the subject has at no time in its long history been subjected to such severe criticism as that of the past decade. A number of its most outspoken critics would stop little short of dismissing it, as a subject of formal instruction, from the elementary course. Others would let it remain but would insist that its materials be drastically modified and reorganized for the purpose of teaching it to elementary children. Still others would severely reduce its content and postpone the study of what remains to the later grades of the elementary school course.

To some of us this rapid growth of criticism is not a little disturbing. Many of us have felt, no doubt, that we were "getting on" in the subject and that soon many if not all of its various difficulties would be overcome. We have pointed with considerable pride to the record of the past twenty-five years—full of surveys, studies, investigations, and experiments which have had for their purpose the improvement of the class-room instruction in arithmetic. Surely these have had an influence for good. At least they have seemed to prove beyond doubt that those interested in the teaching of arithmetic in the elementary school have been keenly aware of the difficulties and have made at least some notable attempts to solve them. Even so, the critics are not as fully impressed as we would like with this array of professional literature to which we point. They seem to prefer to go to the class-rooms where the actual teaching of arithmetic is taking place and to be content to base their criticisms upon their findings. Or they are satisfied to measure at the end of the elementary school course, the final results

* Paper read before the Arithmetic Section of the Conference on "Problems of Elementary Education" at Teachers College, Columbia University on Saturday Morning, January 11, 1935.

of the teaching of arithmetic and upon the basis of these findings to make critical evaluations of the teaching of the subject through the grades. More or less typical of one sort of criticism is that found in the November 1934 issue of the *Illinois Teacher*, in which it is said that "More children fail in school because of arithmetic than because of trouble with any other school subject, with the exception of reading in the early primary classes. Yet arithmetic usually gets the lion's share of the time table and of home work."[1]

Although professional literature on the subject of arithmetic is full of valuable materials on the teaching of the subject, a casual review of its titles impresses one of certain facts. One of the most important of these is the fact that few studies or investigations have been directed at what would seem to be one of the first and most vital problems in the teaching of arithmetic, the professional equipment of the prospective teacher in the field of elementary school arithmetic. The scant recognition which this problem has received in the face of a growing criticism of the teaching of arithmetic, is difficult to explain. Because of this fact and because of his interest in the teaching of arithmetic to elementary school children, and furthermore, because of his experience in teaching professional courses in arithmetic to prospective teachers as well as to employed elementary teachers, the writer was led to undertake the study he has been asked to summarize for you today, a study entitled "The Professional Education of Teachers in the Field of Arithmetic."

In the main the study had a three-fold purpose: first, it attempted to determine what inadequacies, if any, existed in the teacher's knowledge of the fundamental principles of arithmetic which might reasonably be considered to handicap him in his efforts to teach arithmetic effectively to elementary school children; second, it attempted to determine what inadequacies, if any, existed in the teacher's knowledge of effective teaching techniques and in his skill in the application of such techniques in his classroom work in arithmetic; and third, to evaluate the professional courses in arithmetic now offered by professional schools for teachers to students preparing to teach in the elementary schools.

The data for the study were collected from a number of sources. Approximately 20,000 examination papers of candidates for the

[1] Washburne, C. W., "Why Is Arithmetic a Bugbear?" *Illinois Teacher*, Vol. 23, November 1934, p. 69.

license to teach in the elementary schools were sampled and the amount of failure on each of 65 questions dealing with the subject matter of arithmetic, and on each of 75 questions dealing with methods of teaching arithmetic was determined. A special examination of 50 questions dealing with the fundamental principles of arithmetic which underlie its various processes, was given to some 325 teachers who were at the time of the examination employed in the elementary school. Approximately 650 full class-period observations of the teaching of arithmetic to elementary school children by 280 different teachers were made, together with approximately 200 conferences with the teachers whose teaching was observed. The academic and professional education and the professional experience of 37 teachers of professional courses in arithmetic in 24 different professional schools for teachers were reviewed. The instructional materials such as basic texts and reference works used in 51 different professional courses in arithmetic taught by the 37 teachers were reviewed and analyzed. Finally, the class-room activities taking place in these 51 professional courses were determined by a check-list of 450 different activities. The check-list revealed the nature and content of the class-room work and revealed also, by means of a time-unit key, the emphasis or lack of emphasis placed upon this or that activity.

The amount of failure on the examination papers of the candidates for the license to teach was of rather surprising proportions. The 65 questions used on the examinations dealt primarily with the subject matter of arithmetic found in the first six grades of the elementary school. These questions stressed two things, the manipulation of the fundamental processes and the application of the processes to the solution of problems. The failure ranged from 1% made on a question requiring the multiplication of two decimals, to 75% made on a question asking for the answer to a continued example such as $2+4\div2+3\times2$. The average amount of failure was approximately 30%. As was to be expected, failure was less on questions dealing with the manipulation of the mechanical processes, and greatest on questions calling for an application of processes to the solution of problems. Failure mounted rapidly on problems as they become more involved and required the application of two or more of the fundamental processes. When the questions were grouped according to the topics of arithmetic covered and the range and amount of failure on each topic were compared with

those of other topics, the comparison indicated general weakness rather than special weakness on particular topics or phases of the subject.

The amount of failure on the special examinations given to employed teachers was considerably higher. This can be explained by the fact that the 50 questions used dealt primarily with the fundamental principles which underlie the various processes. Failure here ranged from approximately 53% made on a question asking for the reason for setting over each succeeding partial product one place when multiplying a number by a multiplier of two or more digits, to 100% in the case of a question calling for the recognition of the effects on a remainder of the multiplication principle when applied to division of decimals in the solution of problems. The average failure on the 50 questions was approximately 84%. To the writer's mind, these particular data are probably the most significant in this phase of the study, because they indicate decided weakness in the most fundamental phases of arithmetic. It is difficult to imagine teachers who exhibit such inadequacies, being able to teach children an arithmetic that builds fundamental mathematical meanings.

No attempt was made to determine the subject matter inadequacies that might have existed in the knowledge of the teachers whose teaching of arithmetic was observed. However, here and there during the observations and during the conferences which followed, many situations arose which led the writer to have considerable faith in the data already described.

On the side of methods of teaching as applied to the teaching of arithmetic, the amount of failure on the teachers' examinations was slightly higher than that on the subject matter sections of the tests. Although there was little difference in the range of failure, both ranging from 1% to approximately 80%, the average failure on the methods of teaching questions was approximately 41% as against 30% on the subject matter questions. If one is disturbed by the amount of failure represented here, the data secured by the observations of the teaching of arithmetic give little comfort. Before beginning the observations, a list of some 35 generally accepted principles of teaching was made. The observer, with these in mind, attempted to determine to what extent the activities taking place in the class-room indicated that the teacher was attempting to put into practice one or more of the principles as occasions arose which

gave him the opportunity. The skill with which the teacher applied the principle or principles did not enter into the evaluation. Of course not all the teachers had an opportunity to use any considerable number of the principles during any one class period. Certain phases of drill work exhibited a small amount of failure, as for example, of 278 teachers, only 27 or 10% failed to recognize the "practice in error" principle. Failure was high on certain other phases of drill work, as for example, of 278 teachers, 265 or 95% failed to recognize the "maximum response in minimum of time" principle. Other illustrations could be given but suffice it to say that the failure ranged from 8% to 95%. During the conferences following the observations, teachers as a rule gave evidence that their knowledge of acceptable methods or principles of teaching was considerably better than their teaching would lead one to believe. Most of the teaching of arithmetic observed impresses one of the fact that the theory of teaching and the practice of teaching are still quite unrelated.

Before drawing conclusions concerning the implications of the data reviewed so far, let us consider the work of the professional schools from which a majority of these teachers came. Let us begin with the teacher of professional courses in arithmetic. Although speaking in terms of the median or average is unsatisfactory even if at all possible, we shall attempt it here for the sake of brevity. The teacher of professional courses in arithmetic in the professional school completed his elementary education in a rural or small village school and graduated from a small city high school. Leaving high school he attended a normal school or a teachers college and after thirteen years of "shopping around" a little we find him receiving his bachelor's degree from a liberal arts college. Nine years later he received his master's degree from a university. In both his undergraduate and graduate work, he did not specialize to any considerable extent in the field of mathematics and his graduate work was especially confined to a field other than that of mathematics. His professional experience as a teacher did not include experience in the grades for which his professional courses in arithmetic attempt to prepare teachers. In fact the most of his teaching experience was gained in the high school and it was from this type of school he went to his present position in the professional school, a position he has held for the last nine years.

To continue to speak in terms of the average, the teacher of a

professional course in arithmetic uses in his class-room work a single basic text and follows its organization and content quite consistently, thereby making it unnecessary to have detailed outlines or syllabi for himself or for his students. He finds little time for reference work except possibly to refer his students now and then to a reference book similar in content and in organization to that of the basic text used in his class-room work. The 40 hours of class-room time allotted to his course is taken up with a wide range of activities which center around those dealing with methods of teaching arithmetic and around the review of arithmetic on the elementary school level. The topics he selects for consideration and the emphasis he places upon them, fail to agree to any considerable extent with those found in similar professional courses in arithmetic as taught in other professional schools for teachers. During the teaching of the course, he finds no time to take his class to the training school for observation work nor does he have the opportunity to take his class to the training school for the express purpose of demonstrating the theories of teaching he has advocated in his class-room work. Finally he closes the course with an examination whose questions are of the true-false or completion type, 45% of which deal with the subject matter of arithmetic in a way that is not essentially different from that found in the promotion examinations in the four upper grades of the elementary school. The other 55% of the questions of the examination deal with methods of teaching, tests and measurements, psychology of arithmetic, etc.

So much for the general summary of the findings of the study. The general conclusions are as follows:

1. Although elementary school teachers have a fairly adequate command of the various mechanical processes of arithmetic, they are significally lacking in their knowledge of the fundamental principles underlying these processes and are lacking in an adequate ability to apply the processes to the solution of problems. Therefore it would seem that such teachers would be unable to teach arithmetic in any other than a mechanical way, and could make of their students little more than mathematical automatons.

2. Elementary teachers are definitely lacking in their ability to use in their class-room work in arithmetic many of the most fundamental principles of teaching which the professional schools have tried to have established in the practice of teaching in the elemen-

tary grades. The almost complete divorcement of theory from practice that exists in the administration of professional courses in arithmetic may be in part an explanation for such a problem.

3. That the professional school for teachers is responsible to a very considerable extent for the conditions revealed in this study, seems evident from the following:

a. Teachers have been employed to teach or have been given the responsiblity of teaching professional courses in arithmetic who have not definitely prepared to teach the subject. The teachers of such courses seem to have come to their present positions more by accident than by the urge of special and specific preparation, thus following an educational tradition of long standing.

b. The instructional materials used in the various courses are decidedly limited in extent and are used more often than not in quite an elementary way. Since these materials stress methods of teaching and review of arithmetic on the elementary school level, the prospective teacher has little opportunity to vitalize his knowledge of arithmetic, to perfect his skill in the application of its processes, or to acquire a versatility in the subject beyond that attained during his experience as a student in the elementary school. The mathematics he pursued in high school may have contributed to his understanding of the subject matter of arithmetic, but in what ways and in what amounts, no one seems to know.

c. The activities taking place in the class-room where professional courses in arithmetic are being taught, represent a diversity of practice that is little short of bewildering. The whole situation leads one to believe that the aims and objectives of such courses are decidedly varied if not highly confused.

In the interpretation of the data, you no doubt realized that the writer does not hold to certain theories of education. First of all, he does not subscribe to the theory of education that places the subject of arithmetic in the category of "tool subjects." He does not deny that one of the values which must come from the teaching of arithmetic as well as from the teaching of other subjects, is the utilitarian value. But to his mind, this value is or should be quite secondary to a much more fundamental one, the development of the fundamental abilities of children. Just as soon as we teachers begin to make our instruction mechanical or to make it mechanical in any one subject, just then do we begin to cut off the operation of one of the most fundamental urges to com-

plete education with which children are endowed, the urge that insistently asks and searches for the "why" of things. This urge is not stimulated and made to develop by mechanical teaching. Such teaching kills it. What part of the dislike for arithmetic which is so evident among the students as they progress from grade to grade and the dislike shown by a large majority of the teachers of elementary school arithmetic is traceable to a mechanical teaching of the subject? Enthusiasm for things that are never understood has not been a very common human trait. As just one illustration of the effects of the mechanical teaching of arithmetic in the elementary schools, let me cite the following: During the 650 observations which were made for this study, approximately nine thousand children were observed in classes where they were being taught arithmetic. Of this number only two students asked the "why" of the processes that had been taught.

In the second place, he does not subscribe to a psychology that contributes its force largely in the direction of mechanical teaching. Some one has said in this connection that children are not concerned in the "why" but in the "how" of things, especially in their study of arithmetic. If this is describing what is actually the situation concerning the student's attitude toward arithmetic, then the teaching of arithmetic has been an educational failure if not a definite hinderance in the educational process. If it points to what is supposed to be an accepted aim of teaching, then it indicates a considerable lack of experience in the teaching of children on the part of its author.

As a natural outcome of these theories as applied to the teaching of arithmetic, there has come the development of the testing movement which has contributed and continues to contribute its full force towards fixing the hold these theories have on the situation. Probably no one single factor has been more effective in making the teaching and the learning of arithmetic a mechanical affair than the testing movement, supplemented as it is by various kinds of practice materials and urged on by the standardization of mechanical performance.

Finally, in the light of the findings of the study reviewed and the short discussion which followed, what can one reasonably say concerning the question "Are We Teaching Arithmetic Effectively"?

The Congruence Theorems by a New Proof

By H. C. CHRISTOFFERSON
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A RECENT BOOK* on professionalized geometry suggests that the congruence theorems can all be proved by the use of a construction postulate instead of the old superposition method. Aside from the philosophical arguments against superposition and the hypothetical constructions involved in the usual sequence, the chief objection to superposition seems to be a pedagogical one. Every theorem serves two functions: a pattern to be followed in proving other propositions, and a general law or principle which can itself be used in proving other propositions. Whatever value the proof by superposition may have as a pattern proof, must at once be destroyed since it is not a desirable method. In fact most teachers have had the experience of having a pupil try to prove exercises by superposition after he has been subjected to the use of superposition in the congruence theorems. The teacher has then been forced to advocate the abandonment of the superposition technique and the substitution of a technique more generally useful.

It is the purpose of this paper to present a different proof for the congruence theorems, with the hope that such an attempt may provoke some discussion among those interested and stimulate some teachers to experiment in their geometry classes. Among the postulates used or implied in the following proofs eight are listed below.

1. With a given center and radius only one circle can be drawn in a given plane.
2. By means of the compasses a length may be measured off equal to any given line segment.
3. Only one straight line can be drawn between two points.
4. A straight line is the shortest distance between two points.
5. Two straight lines can intersect in only one point.
6. A circle can cut its diameter in only one point on each side of the center.

* Christofferson, "H. C. Geometry Professionalized for Teachers." George Banta Publishing Co., Menasha, Wis., pp. 72–73.

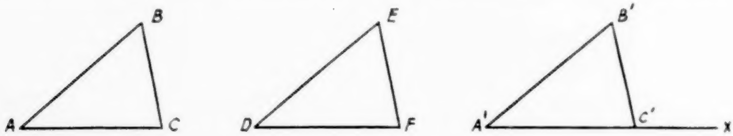
7. If the distance between the centers of two circles is less than the sum of their radii and yet greater than the difference between their radii, the circles will intersect in two points, one on each side of the line joining their centers.

8. The Construction Postulate: If, at a given position on a given line, only one triangle can be constructed by using a given set of conditions, then all triangles which have been constructed from, or which conform to, the given set of conditions are congruent.

Only one of these postulates is new, the last one. The others have all either been used or implied in all geometries. Numbers 2, 6, and 7 are not commonly stated in geometry textbooks, yet they are always implied in all constructions which involve equal segments or intersecting arcs. They are stated here merely for emphasis and approximate completeness.

Congruence by Three Sides

Theorem: Whenever two triangles have three sides of one equal respectively to three sides of the other, the triangles are congruent.



Given: Triangles ABC and DEF with $AB=DE$, $BC=EF$, and $AC=DF$.

To Prove: Triangle ABC is congruent to triangle DEF.

Proof: 1. $AC<AB+BC$. Postulate 4.

2. $AC>AB-BC$ or $BC-AB$ since $AC+BC>AB$ or $AC+AB>AC$. Postulate 4 and the axiom: Equals subtracted from unequals leave unequals in the same order.

3. Lay off $A'C'=AC$, on working line $A'X$. Postulate 2.

4. With A' and C' as centers and with AB and CB respectively as radii describe arcs which intersect above $A'C'$. Call the point of intersection B'. Postulate 7.

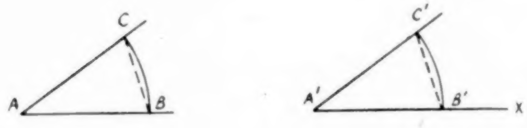
5. Draw $A'B'$ and $B'C'$. Postulate 3.

6. $\therefore \triangle ABC \cong \triangle A'B'C'$. Postulate 8.

7. $\therefore \triangle ABC \cong \triangle DEF$. Postulate 8.

Construction of a Given Angle

Problem: Construct an angle equal to a given angle A.



Procedure: 1. Draw a working line $A'X$.

2. With A as a center and any convenient radius describe an arc cutting the sides of the angle. Call the points of intersection B and C. Postulate 6.

3. With A' as a center and the same radius cut $A'X$ at B' and extend the arc upward. Postulates 6 and 1.

4. With B' as a center and with BC as a radius describe an arc intersecting the arc through B' and call the point of intersection C'. Postulate 7.

5. Then $C'A'B'$ is the required angle equal to angle A.

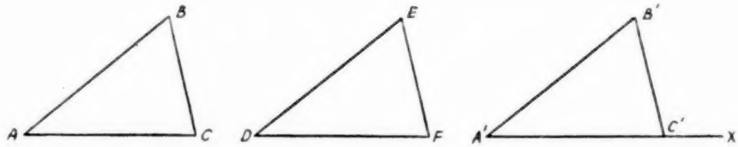
Proof: 1. Draw lines BC and $B'C'$. Postulate 3.

2. Triangles ABC and $A'B'C'$ are congruent by "3 sides."

3. $\angle A = \angle A'$ being corresponding parts of congruent triangles.

Congruence by Side-Angle-Side

Theorem: Whenever two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, then the triangles are congruent.



Given: Triangles ABC and DEF with $AB=DE$, $\angle A = \angle D$ and $AC=DF$.

To Prove: The triangles are congruent.

Proof: 1. Draw working line $A'X$.

2. Construct angle A'=angle A. Construction just proved.

3. Lay off $A'B'=AB$ and $A'C'=AC$. Postulate 2.

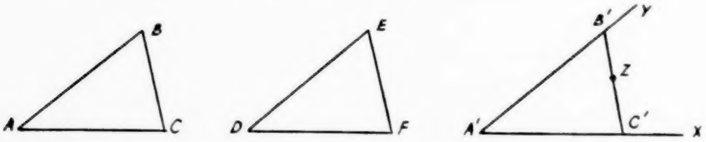
4. Draw line $B'C'$. Postulate 3.

5. Then triangle $ABC \cong$ triangle $A'B'C'$. Postulate 8.

6. Also triangle $ABC \cong$ triangle DEF. Postulate 8.

Congruence by Angle-Side-Angle

Theorem: Whenever two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, then the triangles are congruent.



Given: Triangles ABC and DEF with $\angle A=\angle D$, $AC=DF$ and $\angle = \angle F$.

To Prove: The triangles are congruent.

Procedure: 1. Draw working line $A'X$, lay off $A'C'=AC$, and construct $\angle C'A'Y=\angle A$ and $\angle A'C'Z=\angle C$. Postulate 2 and the construction just proved.

2. Then sides $A'Y$ and $C'Z$ will intersect at some point B'. Postulate 5.

Proof: 1. Therefore triangle $ABC \cong$ triangle $A'B'C'$. Postulate 8.

2. Therefore triangle $ABC \cong$ triangle DEF. Postulate 8.

Note: For greater precision in step 2 above substitute the following for step 2 of the *Procedure*.

2a. Then sides $A'Y$ and $C'Z$ will either intersect in some point above or below $A'C'$ or will be parallel.

2b. Lay off $A'B'=AB$ and draw $B'C'$.

2c. Then triangle $A'B'C' \cong$ triangle ABC by "S.A.S."

2d.. Then $\angle A'C'B'=\angle C$. c.p.c.t.e.

2c. Therefore $\angle A'C'B'=\angle A'C'Z$, since both equal $\angle C$.

2f. Consequently, $C'Z$ intersects $A'Y$ at B'. Postulate 5.

2g. Furthermore, only one angle can be constructed above $A'C'$ as a side and with vertex at C', since the whole is greater than its part.

Conclusion

Thus by means of the construction postulate (8) the three congruence theorems can be readily established early in the course. It is true that like the superposition technique, the technique involved in this proof may not be again used in later proofs, yet the measuring and constructing of lines and angles, and the use of points of

intersection are elements common to most proofs. Furthermore, the criticisms concerning superposition from a logical point of view are avoided. Then too, even if this new proof is used only for the first theorem (three sides), this theorem could be introduced earlier in the course, and many simpler constructions could not only be made but proved. It is always a bit embarrassing to a geometry teacher to start out with an introductory chapter where everything is taken for granted, and then begin in the second chapter to prove everything not postulated or defined. A student is taught in the first chapter to bisect an angle, erect a perpendicular, draw a parallel, and to accept intuitively that his constructions are correct. Then suddenly he is forced to become rigorous. This simple early proof for congruence by "three sides" makes it possible to prove that these constructions are correct in the way that the construction of an angle equal to a given angle was proved in this paper.

Then too in a construction problem there is real need for proof because the pupil may not be entirely sure that the construction really is correct. Such problems provide excellent motivation for demonstration, because after a construction is made there is naturally a "felt need" for proof that the result is correct. In many constructions the proof involves congruence by three sides, and consequently the proof here given which makes it possible to prove congruence by "three sides" first and early in the course makes possible the rigorous proof of all the simple constructions.

The Mathematics Laboratory—A Device for Vitalizing Mathematics

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MATHEMATICS as traditionally taught has met with considerable disfavor among educators and in many cases it has been found to be impracticable. While this may be true evidence does not point to any inherent weakness in mathematics itself. Rather, it has shown that mathematics is a very necessary part of our civilization and, as such, it rightfully deserves a place in the curriculum of our schools. Evidence does point, however, to the fact that the teaching of mathematics is at fault.

Proof of this fact is constantly staring us in the face, especially in times such as these, when the application of mathematics to our taxation problems, our distribution problems, our rehousing problems and producer-consumer problems would give us much information and aid greatly in their solution. The very fact that people in general do nothing about these problems is sufficient evidence that their education has been lacking. It is an argument for the necessity of mathematics and clear evidence of insufficient training in it.

One of the things that we, as mathematics teachers, have failed to do is to show that the same kind of thinking and reasoning that we use to develop mathematical principles may also be used to solve not only our engineering and construction problems but also our social problems.

We have been very much interested in recent years in imparting a certain amount of information that may be used to carry out the activities of the various means of livelihood without understanding why this knowledge was necessary or how new methods might be derived in case the original information proved inadequate. Students leave our class rooms and go into the world to apply their knowledge in a mechanical way but when new occasions arise and their acquired information proves insufficient they are at a loss as to what to do.

Even younger pupils have formed this habit and tell us that if

we will only show them the solution or give them the formula they will work the problem. If we teach mathematics in such a manner, is there any wonder that it has been impracticable? People do not know the formulas to apply and there is no one to advise them. If the mathematics of the secondary schools does not aid the pupils in finding a way to solve their own problems in life, it has failed at least in part. If, however, this objective is to be realized it will challenge every possible bit of initiative, imagination and resourcefulness that a teacher may have.

The day has come when teachers of secondary mathematics can no longer command the attention, interest and reasoning ability of students in an abstract, unexperiential manner. The mathematics laboratory, different from the ordinary class room—not a common work shop, but an awe inspiring environment which bubbles over with the spirit of mathematical science—will aid greatly in commanding their attention and maintaining their interest. It is a laboratory where pupils work, reason and think in a way inspired by the great mathematicians of the past and present, by great achievements, discoveries, inventions and epochs in civilization made possible by mathematics; where they are inspired by the realization of the mathematical nature of the universe making it possible for man to control and alter it to the greater satisfaction of his wants and needs.

The picture thus painted is philosophic and idealistic and, very likely, in the minds of many is the fantasy of a "dreamer"—something unattainable, impracticable and not possible of appreciation by pupils of the junior and senior high school level. Yet there is a keen desire of pupils of these levels to feel that their work is important, that it is a part of the great scheme of life. Nothing gives them more joy than to know that they are working with something vital in the makeup of civilization. As soon as they lose sight of the importance of their tasks the interest lags. True, some have a wider outlook; others look only to the immediate vocations for which they expect to prepare. But even these pupils have a secret feeling that their work plays an important part in the world of affairs.

We can profit greatly by observing these same pupils at work under different conditions. The following incident is an example. A boy who took no interest at all in the Pythagorean theorem when it was studied in class found later, in his *Boy Scout Manual*, that he could apply the problem in doing his map work. Immediately he

tackled the problem with new enthusiasm and, as a result, mastered it and greatly appreciated its use. Anyone who is familiar with the methods employed by the scout organization recognizes the wonderful appeal to boy nature. One of the characteristic features of the method is that of giving the boy a purpose for everything he does. The scout master presents the appeal and the scout pursues the activity of his own free will. Here is an interest and an enthusiasm of which the school has not been able to avail itself.

The appeal made by the art work and industrial education is equally challenging. Recently while I was taking a group of pupils through a large art museum and school, several noticed the symbols $\sqrt{2}$ and $\sqrt{5}$ along certain lines in the architectural drawings of fine buildings. The pupils listened intently and seemed pleased that they had discovered a new use for their geometry. Later one student confessed that he had never realized that geometry was so important.

This same class took an excursion through a large industrial plant with the sole purpose of noticing the different kinds of geometrical figures and designs that they could find. Upon return we tried to list our findings. During this process one boy raised the question of why one type of figure was used instead of another. He felt that a different choice would have been better. A discussion of the advantages and disadvantages of a possible change in construction brought out more geometry than several ordinary class exercises.

These illustrations are cited, not to minimize the work of the school, but to illustrate the fact that the school has not as yet exhausted all of the available resources. Teachers of mathematics fall short of their opportunity by falling in line with a particular educational "fad" and not taking advantage of all the resources at hand. Therefore let it be plainly understood that a laboratory method is not being advocated as a substitute for all present day methods. It is not always a question of substituting one method for another. It is a question of making the best possible use of all sound methods and modes of teaching which are available.

The teacher of mathematics who has a genuine appreciation of the values of the subject will have a desire to arouse the enthusiasm and interest of the students, to hold their attention and to inspire them to the realization of their fullest efforts in order that they too

may gain some insight into the part that mathematics has played in civilization and, hence, in the lives which they live.

It has been stated that the means by which this may be accomplished is by the use of a laboratory so furnished and equipped with instruments and devices that it is possible for pupils to see the part that mathematics has played in changing civilization—its social and its practical value, its part in discovery, invention and construction, its basic foundation for science, and its application as a tool in thinking and reasoning, and remaking society for the good of human kind.

Every room in which mathematics is taught should be of sufficient size, properly ventilated and lighted, to accommodate about thirty-five pupils. It is necessary in some instances to have more than this number in one class but it is not desirable. Classes should average in size somewhere between thirty and thirty-five pupils. Plain desks or preferably adjustable tables and chairs provide the most practical equipment for a room since they may be adjusted to suit the needs of the class.

It is needless to say that plenty of blackboard space is essential. The blackboard work will be far more interesting and, very probably, far more beneficial if the proper blackboard instruments are available. That is, there should be compasses, rulers, protractors, colored crayons and blackboard pointers. Thus, with proper instruments, work at the board may be made truly in accord with mathematical principles of accuracy, neatness, precision and logic. Before leaving the topic of blackboards it must be brought out that one section should be marked in squares for the study of graphs.

A bulletin board is an indispensable item in the equipment of a mathematics class room. Here items from newspapers and magazines which apply specifically to the unit in hand, reference to articles of interest, books and applications of all kinds may be posted. Part of the space may be reserved for a graph showing the average pupil achievement in the units of work, announcements and reports of project work. Another part may be reserved for mathematical recreations of all kinds. Some scheme of advertising these may work out to a great advantage. Yet another section may be reserved for a special project such as "The Geometry of Nature" in which pictures and illustrations contributing to this subject are posted.

Many pictures and charts (especially graphs) showing the application of mathematics in the present day industrial, business and social world are available and should be displayed in an artistic manner throughout the room. Charts showing the symmetry of architectural design are especially interesting to a class of geometry students. Others showing parabolic and other curves as well as orbits of planets and motions of nebulae along with explanations of the mathematical applications is not only interesting but quite desirable as a means of showing the use of mathematics.

Every mathematics room should have at least a few good pictures of mathematicians who have made great advances and about whom pupils would like to learn. Some of these men have interesting biographies and their lives may become an inspiration to all who read them. Their struggles and heroic attempts to advance knowledge in the face of existing superstition makes them heroes in the eyes of the pupils.

A selected group of good books on historical mathematics, reference books and books on mathematical recreations and plays should be available for the use of the pupils at all times. This library need not be large and may consist of only those books which will prove most profitable for classroom work as the school library will be equipped with a more general supply of books for out of class assignments.

Reference materials, projects, reports, notes and papers bearing upon the subject studied should be carefully classified and filed in order that they may be recalled whenever occasion arises for their use. It is suggested that a card catalog of all articles and books in the school library that are especially valuable for class needs be kept in the room for reference in assigning reports and supplementary topics.

Finally there is a list of miscellaneous instruments some of which are indispensable and others which may be added from time to time as the teachers sees fit. Squared paper, cardboard, triangles of thirty, forty-five and sixty degrees, models, string, scissors, carpenter's square, balances, verniers, micrometers, thermometers, angle mirror, hypsometer, clinometer, slide rule, transit, plane table, sextant, level and rod, and other instruments for measuring and surveying are articles which may be used to great advantage by a good teacher. Often times earlier instruments, such as, the cross-staff, astrolabe, and others are sources of much interest as fore-

runners of modern equipment. Any other instruments or devices which may be used to illustrate some mathematical principle or show its application and lend itself to experiment by the pupils should be obtained.

Experience has taught the writer that pupils are very much interested in these devices and they tend to maintain interest in mathematics as experience with such devices increases. However, the mere fact that the proper equipment is available and at hand does not tell the whole story. If not skillfully handled by a teacher of insight and originality, these devices may become merely deadwood and an added expense to the school board.

Above all things those who are responsible for arranging and equipping a mathematics laboratory in addition to seeing the practical side must also have a keen sense of a few of the most important mathematical principles—those of proportion, symmetry, neatness of design, efficiency and accuracy. One can very easily defeat his own purpose by allowing himself to become careless and negligent in the way in which he keeps the room. Nothing could be more hazardous to the whole enterprise than to allow this equipment (with all of its possibilities) to become the source of failure. The good teacher will always be on the lookout for anything that may hinder or stand in the way of the greatest possible achievement as well as those things which tend toward the promotion of this achievement.

OXFORD MEETING

World Federation of Education Associations

August 10-17, 1935

The 1935 meeting of the World Federation of Education Associations will be held, August 10-17, in Oxford, England. The meeting will be in conjunction with two other strong international organizations, the International Federation of Teachers Associations (elementary) and the International Federation of Associations of Teachers in Secondary Schools, which guarantees representation from most of the countries. Persons who attend will have opportunity for making interesting contacts with many lands.

The British Committee is preparing for various sorts of entertainment and special tours under the direction of local committees. There will also be an exhibit of work of children of the English schools. Oxford itself, as the seat of one of the oldest and best known universities, presents many attractions. Distinguished speakers will be on the program and personalities known world-wide will be presented at the general sessions.

The Calculation of the Approximate Rate of Interest in Consumer Credit

By Lucien B. Kinney
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As one phase, perhaps, of the interest that has been developed in consumer education, some of the more familiar aspects of consumer credit are beginning to receive attention both in junior high school mathematics and in commercial mathematics. Many of the modern text books in these subjects include topics on installment buying and building and loan associations, and it is to be expected that the general interest in consumer credit will lead eventually to a rounded treatment so far as this is possible on the secondary school level.

One of the most interesting and instructive types of problem in this field requires the calculation of the cost of buying or borrowing on the installment plan. When we find that in normal times the government can borrow for 3 per cent, a large corporation for $4\frac{1}{2}$ per cent, and a merchant for 6 per cent, while the consumer may pay as much as 300 per cent, we have a fair basis for comparing the cost of consumer credit with that of investment and commercial credit, provided that a sound method has been used for calculating the interest rates paid by the consumer. In treatments of this topic in text books and elsewhere, however, the writer has found a surprising variety of methods in use, leading to different results. When, as is commonly the case, the loan is paid in equal installments, the rate may, of course, be calculated with annuity tables. For the use of high school pupils, however, a simple method of approximation is desirable. It should be based on assumptions that are in accord with business practice, and should yield a result reasonably close to the actual rate.

With these criteria in mind, it is interesting to note the results of a few of the approximations that have been used, when used in a problem, considering the assumptions on which they are based, and comparing the interest rates found by the approximations

with the rates as calculated with annuities. We may take this problem as an example:

The cash price of an electric sewing machine is $133. The terms on the installment plan are: $25 down, and $9 a month for 14 months. Find the annual rate of interest on the unpaid balance.

It is common practice in this type of problem to consider the entire carrying charge as an interest charge. There is some difference of opinion on this point, and there is no question but that the pupil should be aware of the expenses that make up the carrying charge from the merchant's point of view. From the standpoint of the consumer, however, it is not incorrect to consider the difference between the cash price and the installment price as an interest charge. In each of the solutions to be illustrated the difference of $18 is taken as the interest.

Solution 1.

Total price by installment plan	$151.00
Cash price	133.00
Interest	$ 18.00

The principal is 14 payments of $9 each, payable in 1, 2, . . . 14 months, a total equivalent to $9 for 105 months.

To find the rate of interest, $r = \$18 \div (\$9 \times \frac{105}{12}) = 22.9\%$.

Comment. The "14 payments of $9 each" do not comprise the principal. They are payments both of interest and principal, since they include the $18 interest charge. 22.9 per cent, therefore, is an approximation, not of the rate of interest, but of the rate of discount. It is unfortunate that this solution is commonly found not only in junior high school text books in mathematics, but even in some of the recent commercial arithmetic text books. If the authors of the text books in commercial arithmetic in particular had provided for the use of amortization schedules in introducing such topics as this, they would have avoided the error of using this method.

Solution 2. The principal for the first month is the balance of the cash price after the down payment is made, thus:

Cash price	$133.00
Down payment	25.00
Balance	$108.00

This balance is paid in the first twelve installments, the remaining two installments being considered as payments of interest. Hence,

with this method, the term for which credit is extended is twelve months, and \$18 is the interest for one year.

Since the monthly balances have a common difference, the average balance is the average of the first and last monthly balances:

$$\frac{\$108+\$9}{2}=\$58.50.$$

The annual rate of interest is therefore \$18÷\$58.50=30.8 per cent.

Comment. Since it is assumed that the interest is all paid at the end of the term, this is an approximation of the effective rate of interest. While the distinction between interest and principal is recognized in this approximation, the assumption that the borrower has two months in which to pay the interest after the term of credit has expired would represent, in actual practice, an unusual arrangement. Furthermore, while the method is simple in the case illustrated, since the balance is paid in an integral number of installments, it becomes rather awkward to interpret when a fraction occurs.

Solution 3. The first principal and the interest charge, as before, are \$108 and \$18 respectively. It is assumed, however, that each of the 14 payments is part principal and part interest. On the average, then, each installment of \$9 is made up as follows:

Payment on principal (1/14 of \$108)	\$7.71
Interest (remainder of the \$9 payment)	1.29
Total monthly payment	\$9.00

As in the previous solution, the average balance is the average of the first and last monthly balances; though the figures are different:

$$\frac{\$108.00+\$7.71}{2}=\$57.86.$$

Interest per year is $\$1.29\times 12=\$15.48$.

Annual interest rate is \$15.48÷\$57.86=26.6 per cent compounded monthly.

Comment. Since it is assumed that each payment is part interest, this is an approximation of the nominal interest rate, compounded at each payment.

Solution 4. As a standard for comparison, we may take the interest rate as found by using an annuity table. Since a debt of \$108 is to be extinguished by 14 payments of \$9 each,

$$\$108 = 9(a_{\overline{14}|} \text{ at } i)$$

Whence $i = 2.13\%$ per month, or 25.6% per year compounded monthly. The effective rate is 28.8% annually.

Summarizing, the rates found by the four methods are:

Solution 1	22.9%
Solution 2	30.8%
Solution 3	26.6%
Solution 4, nominal	25.6%
Solution 4, effective	28.8%

When the method of solution 1 is used the rate that is found is always lower than the actual effective rate, on account of the assumptions on which it is based. The method of solution 2, on the other hand, is based on assumptions that cause the indicated rate to be a little above the actual effective rate. The approximation used in solution 3 yields a rate that is always close to, and slightly above, the actual nominal rate. Experience shows that the difference is commonly proportional to that found in this case—about 1/25 of the actual rate. Both from point of view of accuracy of results, and business usage, it appears that the third approximation is superior to the other two.

Gaspard Monge

Born at Beaune, May 10, 1746
Died in Paris, July 28, 1818

THE MATHEMATICAL work of Gaspard Monge is of too advanced a character to be included in this series of biographical notes, but the other details of his career present many points of interest to the high school student. The account which follows is based chiefly on the histories of Ball and Cajori, and on Professor D. E. Smith's essay "Gaspard Monge, Politician" which appeared in *Scripta Mathematica*, and which was reprinted in *The Poetry of Mathematics and Other Essays, Scripta Mathematica Library*, Vol. 1.

Jacques Monge, the father of the famous mathematician, seems to have been a laborer, perhaps a peddler, and according to Arago, he was scissors-grinder in Beaune as well. Each of his three sons showed ability in mathematics. Louis became professor of mathematics in the Ecole royale militaire in Paris and in the Ecole royale de Génie at Mézières. Jean Baptiste was professor of mathematics in the Ecole militaire in Rebais and later he was professor of hydrography at Antwerp. Jacques Monge sent each of his sons to the collège of the Congrégation de l'Oratoire at Beaune. Professor Smith notes that Gaspard Monge received the first prize in all of his work and was characterized as "puer aureus." At sixteen, Gaspard Monge was given the chair of physics in the collège which the Oratorians maintained at Lyons. It is said that his drawing of the plan of the town of Beaune won him the notice of an army officer who procured his admission to the military school at Mézières. Ball says that

> His birth, however, precluded his receiving a commission in the army, but his attendance at an annexe of the school where surveying and drawing were taught was tolerated, though he was told that he was not sufficiently well born to be allowed to attempt problems which required calculation. At last his opportunity came. A plan of a fortress having to be drawn from data provided by certain observations, he did it by a geometrical construction. At first the officer in charge refused to receive it, because etiquette required not less than a certain time should be used in making such drawings, but the superiority of the method over that then taught was so obvious that it was accepted; and in 1768 Monge was made professor, on the understanding that the results of his descriptive geometry were to be a military secret confined to officers above a certain rank.*

* W. W. R. Ball, *Short Account of the History of Mathematics*, 1915 ed., p. 426.

Even more than this, the secret was to be kept to the school at Mézières, the jealousy between the military schools being too great to allow of its publication. Arago described this work as "the brightest jewel in the crown of our confrère."

At Mézières, Monge began the work in the equations of surfaces which called forth the comment from Lagrange "Avec son application de l'analyse à la répresentation des surfaces, ce diable d'homme sera immortel!"

In 1780, Monge was given the chair of mathematics in Paris where his brother was already living. Here, a person is tempted to raise a query:—Napoleon Bonaparte graduated from the Ecole royale militaire in Paris in 1785. Assuming that Louis Monge was professor of mathematics in that institution for several years after 1780 when we know he was there, is it not possible that the friendship between Bonaparte and Gaspard Monge may not be traced in the first instance to this fact?

Friendship with Condorcet, led to the appointment of Monge as professor of hydraulics and later to his election to the Académie. In 1783, Monge was made examiner of candidates for the navy. Meantime, however, his descriptive geometry was kept a secret. Cajori quotes a conversation which took place in 1780 in which Monge is said to have remarked to Lacroix and another student, "All that I have done here by calculation, I could have done with the ruler and compasses, but I am not allowed to reveal these secrets to you."

Louis and Gaspard Monge joined the revolutionists in 1789, and Monge had an important part in the development of the metric system. From August 1792 to April 1793, he was Minister of the Navy and Colonies. During the Terror, he was obliged to flee from Paris, but he was recalled in 1794 to teach at the Ecole normale. At this time his descriptive geometry was published. He later taught at the Ecole polytechnique.

In 1796, he was sent to Italy on the wretched business of collecting the works of art that were being levied from the various cities in lieu of war contributions in cash.

Monge seems to have been close to Bonaparte in his plans for the Egyptian expedition. Dr. Smith records his activities in these terms,

In May, 1798, he joined the fleet, and on June 9 was at Malta, entering the capital with Napoleon three days later. Neither of these men was guilty of pro-

crastinating. Within twenty-four hours, under the direction of Monge, arrangements had been made for fifteen primary schools and one central school and the course of study was ready for use. . . .

In August, Napoleon created the Institut Egyptien in which the section of the mathematical sciences included Napoleon himself, and Monge, Fourier, and Malus. Bonaparte was urged by his confrères to accept the presidency, but he declined, saying: "Il faut placer Monge, et non pas moi, à la tête de l'Institut; cela paraîtra en Europe beaucoup plus raisonable.*

Monge escaped to France after the Battle of the Nile. When Napoleon was made Emperor, Monge was given the title of Comte de Péluze. He was with Napoleon during at least a part of the Austrian Campaign. On the fall of Napoleon, Monge lost his honors and offices and he died in poverty.

VERA SANFORD

* David Eugene Smith, *The Poetry of Mathematics*, New York, 1934, pp. 83–85.

Abstracts of Recent Articles on Mathematical Topics in Other Periodicals

By NATHAN LAZAR

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Algebra

1. Candy, A. L. *To construct a magic square of order 2 n from a given square of order n*. National Mathematics Magazine. 9: 99–105. January 1935.

The second and final installment of a paper, the first part of which was published in the April 1934 issue of the magazine then known as The Mathematics News Letter.

2. Muirhead, R. F. *Note on the decimal point*. The Mathematical Gazette. (London, England) 19: 42–43. February 1935.

The author is very much concerned with the fact that the present position of the decimal—to the right of the units' place—makes the rule for the positive characteristic different from that for the negative one. If the units' place and *not* the decimal point be taken as a starting point, one rule suffices for both. "Characteristic equals the number of steps to the right from the leading digit to the units' place. (Steps to the left being reckoned negative.)"

The writer suggests at first that the decimal point be placed below the units' place. But this is rejected on account of its awkwardness for the printer. "Probably the easiest change to make would be to have *two* decimal places, one on either side of the units' place. This would restore symmetry and give no trouble to the printer."

The writer has evidently forgotten the amount of trouble his suggestion would give to the teachers who find it difficult to acquaint their pupils with the mysterious behavior of *one* decimal point. There are, moreover, so many other difficulties that would arise as a result of the introduction of the double decimal point that one wonders whether it will be worth all that trouble just for the sake of facilitating the remembering of *one* simple rule in logarithms.

The second suggestion is, however, worthy of serious consideration by teachers of mathematics, i.e., that we follow the practice of Lord Kelvin of writing a large or small number in the form of (a number lying between 1 and 10)$\times$(a power of 10). Thus 3,520,000 is written 3.52×10^6, and .0000000178 $\equiv 1.78\times 10^{-8}$.

The frequent occurrence of the recommended form in current text books and scientific literature is sufficient argument for its inclusion in, at least, a second course in algebra.

It is of interest to note, in this connection, that that topic is already included in the syllabus of at least one state department of education (N.Y.) as well as in many American text books in Intermediate Algebra.

3. Richert, D. H. *Concerning I^n*. School Science and Mathematics. 35: 191. February, 1935.

The writer proposes another method for finding the value and the sign of I^n, which he believes is simpler than the one proposed by R. L. Posey in the Novem-

ber issue of SSM. (The review in this department in the issue for January 1935, contained an obvious typographical error.)

To find the sign and value of I^n write n in the form of $4m+k$ which is always possible. m is any one of the infinite set $0, 1, 2, 3, \cdots$ and k is one of the finite set 0, 1, 2, 3.

If $k=0$, then $I^n=1$.
If $k=1$, then $I^n=i$.
If $k=2$, then $I^n=-1$.
If $k=3$, then $I^n=-i$.

Many teachers will recognize in the method outlined above a procedure which they have followed themselves without having analyzed it algebraically. It corresponds to the working rule which they have probably given to their pupils, without a proof.

(a) Divide the exponent of n by 4 and examine the remainder.
(b) If there is no remainder, the value of I^n is 1, etc.

4. Smith, C. D. *On the algebra of mixtures.* National Mathematics Magazine. 9: 138–41. February 1935.

A generalized treatment of this ever interesting application of algebra, with particular attention to the criteria of solubility of problems. Since the treatment assumes no knowledge beyond determinants, it falls within the range of high school students of advanced algebra.

5. Sykes, Mabel. *Some criticisms of recent ninth grade algebra texts.* School Science and Mathematics. 35: 153–60. February 1935.

The discussion is based upon ten books with the following copyrights: one of 1928, two of 1929, three of 1932, two of 1933 and two of 1934. The books were studied not only with respect to content, but also with respect to order of treatment, provisions for individual differences, modern type objective tests, systematic instruction in problem solving and diagrams and pictures.

"In general it is evident that authors of recent books appreciate to some degree at least the necessities of present day conditions. One wonders however if the books are advancing as fast as even the conservatism of schools and teachers would permit."

Arithmetic

1. Andrews, F. Emerson. *Revolving Numbers.* Atlantic Monthly. 155: 208–11. February 1935.

Teachers of mathematics will be glad to read the following comment by the editors of the Atlantic Monthly. "When we made the venture of 'An Excursion in Numbers' last October [reviewed in this department in December—, N.L.], we hoped, at best, for a limited group of appreciative readers, but found instead that x = Infinity. In response to demand, F. Emerson Andrews, manager of publications for the Russell Sage Foundation sends us a mathematical sequel, 'Revolving Numbers.'"

This article contains a popular account of the well known phenomenon in the realm of numbers, known as "periodic fractions" or "repeating decimals"

An example of a revolving number is 142857. "Multiply it by 2 it becomes 285714—the same digits, but commencing at a different place in the series. Multiplied by 3 it is 428571—again the same digits, but commencing at a still different place. Multiplied by 4, the answer is 571428, still the same series." Multiplied by 5, it becomes 714285, and by 6 it is 857142. But here the charm ends; for multiplied by 7, the original number yields 999999.

Another example of a revolving number—and they are infinite—is 0588-235294117674. If multiplied by any number from 2 to 16 it will always re-

produce revolving forms of the original number. The author finally lets the cat out of the bag by saying "that all the revolving numbers we have so far examined, and presumably all the existing numbers with similar properties, are in reality repeating decimals . . . which are always produced when unity is divided by a prime number, excepting only 2 and 5, which, though prime are also divisible into our number base 10." The repeating decimals for $\frac{1}{7}$ is .142857 142857 . . . which produced our first revolving number, and the repeating decimal for 1/17 is the second number quoted above.

It is especially interesting to observe that the revolving numbers are *not* the product of decimal counting—the basing of a number system upon 10; but that any number system which employs a zero produces a similar series of revolving numbers. The author exhibits a revolving number written in the *duodecimal* notation, consisting of 102 places, which he claims, will reproduce a revolving form of the original number if multiplied by all numbers from 2 to 102!

As a result of considerable experimentation in several number systems, a set of six rules is announced. A logical explanation, involving no mathematics, is also included as an aid in the understanding of the curious phenomenon—revolving numbers.

2. Bellet. *Procédé simple pour trouver mentalement le jour d'une date quelconque du calendrier*. Revue générale des sciences pures et appliquées 45: 473–74. August 1934.

Description of a simple method of finding the day of the week of any date in the calendar, past or future.

3. Kinney, Lucien B. (a) *Aims and content in commercial arithmetic*. Journal of Business Education. November 1934.
   (b) *Teaching the fundamental skills in commercial arithmetic*. December 1934.
   (c) *Problem solving and the language of percentage*. January 1935.
   (d) *Consumer education and commercial arithmetic*. February 1935.

A series of four articles on various phases of the teaching of business arithmetic. A bibliography is appended to each article.

4. Stelson, H. E. *Computation of the simple interest rate in installment purchases*. National Mathematics Magazine. 9: 135–38. February 1935.

"Although it is well known that the simple interest rate is smaller than the compound interest rate for a series of payments on a given date extending over periods less than one year, the magnitude of this difference is not usually recognized. Moreover, in view of the current practice of charging simple interest in transactions in which the time is less than a year it seems more logical to inquire the simple interest rate than the compound interest rate.

"It is the purpose of this paper to develop some approximation formulas for computing the simple interest rate."

References are included.

5. Wilson, Dorothy. *Mathematics and home economics*. The School (Toronto, Canada) 22: 418–23. January 1934.

An interesting and detailed description of a few of the more simple problems, which show how arithmetic can be related to problems of home economics.

Geometry

1. d'Ocagne, Maurice. *Nouvelles et curieuses constructions géometriques approchées*. (*Some new and curious approximate geometrical constructions.*)

Revue générale des sciences pures et appliquées. 45: 321–22. June 15, 1934.

(a) The author describes two methods of inscribing a regular nonagon in a circle, which were communicated to him by the inventor, Cordilha, of Rio de Janeiro. The first method yields a side which is theoretically accurate within .0000017 of the radius. The second method is correct to -0.000000002 of the radius.

(b) A method of trisecting an angle is described, which was reported in this department in the issue of March 1935, in another connection.

It should be emphasized that the constructions described above are performed within the platonic limitations of compasses and straight edge, but do *not* yield strict theoretical proofs. The author calls such constructions "construction approchée normale."

Trigonometry

1. Byrne, F. Emerson. *Vector analysis and trigonometry.* National Mathematics Magazine. 9: 130–34. February 1935.

"It is the object of this article to put into evidence some discussion of trigonometry which belongs to vector analysis rather than to school geometry."

2. Richardson, L. *Simple proofs for* sin $(A \pm B)$, cos $(A \pm B)$. School Science and Mathematics. 35: 175–77. February 1935.

Interesting and simple proofs of the well known formulas in trigonometry for those students who possess a knowledge of analytical geometry. For obvious reasons these proofs could not lighten the burden of the teacher of trigonometry in the high school or even at college.

Miscellaneous

1. Byrns, Ruth, and Henmon V. A. C. *Entrance requirements and college success.* School and Society. 41: 101–04. January 19, 1935.

"It was the purpose of this study to control the factor of intelligence and to determine the relation between the college achievement and amount of high school mathematics and foreign language, when this factor was eliminated."

With the help of a very elaborate statistical apparatus, the authors come to the following conclusion—"The pragmatic sanction may justify the belief that foreign languages and mathematics are valuable instruments of instruction, but the evidence does not indicate that they are sacrosanct. They have probably been valued out of proportion to their significance for success in college. Their prominence both in the high school curriculum and in college entrance requirements must be justified on some grounds other than mental training, for the facts seem to show that these subjects do not develop a student's capacity for successful college work."

There is no need to repeat in the columns of this journal the arguments for the study of mathematics. But one reader at least does not recall ever reading in the writings of a recognized, contemporary mathematical educator that mathematics should be taught for the general mental training it affords, or for its development of a student's capacity for successful college work.

No, we are not so naîve as to believe that the study of mathematics *automatically* improves the capacity of the student to reason well in all other fields of thought. But we do contend—and experiments are in progress now substantiating the belief—that mathematics *could be taught* so as to give to many other subjects a maximum of transfer of those habits acquired in the rigor of its reasoning and in the analysis of underlying assumptions. (See below

Wright, H. A. Mathematics in the liberal arts college.)

2. Cairns, W. D. *Advanced preparatory mathematics in England, France and Italy.* American Mathematical Monthly. 42: 17–34. January 1935.

An exhaustive study of the kind of mathematical training received by the students of England, France and Italy in those years that correspond to our lower college courses. Samples of examination questions, lists of text books used and representative syllabi are included. This article contains a fascinating array of contemporary historical material that should be read by every teacher of mathematics. (See below article by Watkeys, C. W. Discussion of the Cairns report.)

3. Colpitts, Julia T. *Mathematics in Japan and China.* National Mathematics Magazine. 9: 123–29. February 1935.

A lively and informative first hand account of the teaching of mathematics in contemporary China and Japan.

The writer describes with infectious enthusiasm and in great detail the text books, the methods, the general educational scheme, the curricula, the entrance requirements, the script used, the universality of the abacus and even some of the problems taught.

4. Ellis, Charles A. *Mathematics and engineering education.* School Science and Mathematics. 35: 123–32. February 1935.

The reflections of a professor of structural engineering on the relation between mathematics and engineering.

It is interesting to discover that Galileo, Coulomb and Euler have been illustrious engineers as well as famous pure scientists. But the advice given to teachers of mathematics will not be received so kindly. Very few will subscribe to the following credo. "I hope for his [the pupil's] sake that you insist that he complete the square in every exercise he does, for, to me it seems entirely unnecessary and even sinful [!] to require or suggest to a student that he memorize a queer arrangement of letters and signs, when all he needs to remember is a sentence of three words, Complete the square!"

Nor will progressive teachers of geometry relish the following *obiter dictum.* "Since the processes of thought employed in the proof of a geometrical proposition do not parallel those employed by the physical scientist or the engineer, we must look elsewhere if we are to find the true value of geometry to science or more particularly to engineering. This value is found in the knowledge of geometrical truths which result from proofs rather than in the proofs themselves." We shall not be so unkind as to point out that the position taken here contradicts the one taken above in respect to the "sinful" use of the quadratic formula.

Perhaps it is unsportslike to find fault with those who so graciously come from other fields to talk to us; but is it too much to ask that they at least acquaint themselves somewhat with our problems and ideals before advising us in our calling?

5. Leib, David D. *The influence of science on contemporary thought-mathematics.* Report of the New England Association of Chemistry Teachers. 36: 30–37. December 1934.

This article is one of three in a symposium on the influence of science on contemporary thought; the other two deal with the physical and biological sciences.

Evidently the recent attack on the importance of mathematics is bearing

fruit—a rallying to the defence of the most ancient and yet the most potent and fruitful of the sciences. "Recently the idea of mathematics as a lifeless tool, a conglomeration of rules and formulas, a mass of facts from which each person should select on a basis of utility, has tended to emasculate the subject in high school and to a degree in college. No one group has contributed to this pernicious doctrine as much as some of our 'educators,' leaders of our schools of education, and professors of education. It still remains true that if our purpose is to stimulate students to really think, there is no tool so powerful for expression, no challenge so forceful to real thought as that of mathematics."

Many appropriate quotations from Emerson, Keyser and Weyl are included as well as a suggestive discussion of the concepts of infinity, invariance and probability. The paradoxical relation between pure science and concrete application is clearly brought out and explained. "The mere fact that mathematics may be tending to become more and more abstract does not mean that it *ipso facto* is less and less applicable to facts. The direct opposite is true. As A. N. Whitehead says 'The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thoughts of concrete facts.'"

It is regrettable that so excellent an article will not reach the wider audience it deserves because of its publication in an organ of relatively local circulation.

6. Neville, E. H. *The food of the Gods.* The Mathematical Gazette. 19: 5–17. February 1935.

The presidential address to the Mathematical Association of England, January 1935. The thesis set forth is "that changes in emphasis in creative mathematics, which have now a direct influence on teaching at the university, ought to have a greater and far more rapid influence on the teaching at the school than they seem to have."

7. Seidlin, Joseph. *Analysis and interpretation of survey of present status of secondary mathematics in the United States.* National Mathematics Magazine. 9: 143–46. February 1935.

A study of the replies received from the State Superintendents of Education to the following three questions:

(a) Has mathematics as a required subject for graduation from high school been eliminated in your state?

(b) If it has not been eliminated as a requirement, is there a definite prospect that it will be eliminated at an early date?

(c) In your judgment, are there good reasons for the hope that mathematics will have increased use in the secondary schools of your state?

A table is included which contains, (a) a list of the states that require at least one year of mathematics beyond the eighth grade, (b) a list of those states that require no mathematics beyond the eighth grade, and also (c) the approximate population of the states mentioned.

"We find that 31 states require at least one year of high school mathematics; 16 states require no mathematics beyond the eighth grade. Six of the former may drop the requirement in the near future; only one of the latter indicates an increase in the use of courses in mathematics. We may reasonably expect, then, that within a year or two the high schools of half the population of the country will be free from state required courses in mathematics beyond the eighth grade."

Another conclusion that the writer feels safe to reach is that "courses in algebra and geometry as state required

courses for graduation from high school, are losing ground. This may be due to poor teaching. It is extremely doubtful, however, that poor teaching is monopolized by teachers of algebra and geometry. The 'depression' may have something to do with it. Or it may be that the present reigning power of educationalists are 'economics' and 'culture' minded, even as a generation ago they were 'science' minded. We do have our fads even in the principles and philosophies of education. . . .

"Unfortunately there are those in position of authority and influence in the framing of educational policies who are prejudiced against mathematics. In most cases the prejudice dates back to some sad incident in their own first contacts with courses in mathematics. In other words, their prejudice is not so much against a great field of human learning as against some unfortunate element in their own experience with it, perhaps the way it was taught (to them). What they propose, however, is incredibly illogical. It is like eliminating water from the diet of humans because some water is contaminated. Obviously, purifying the contaminated water would be the more logical and certainly the more health-benefitting procedure.

"But perhaps, it is too much to expect logical thinking from people who, upon their own admission, have had little training in logical thinking. How about ourselves—teachers of mathematics?

"According to this survey, mathematics is still among the required subjects in most of the secondary schools of the country. Shall we merely sit back and watch courses in mathematics thin out and disappear; then jointly mourn in their passing? Or shall we bend our joint efforts towards the intrinsic improvement of both content and the methods of teaching of courses in mathematics from the second grade on, so that even the legislators of educational policies may perceive clearly the necessity for training in mathematics as constant through all primary and secondary schooling?"

8. Watkeys, C. W. *Discussion of the Cairns report.* American Mathematical Monthly. 42: 34–37. January 1935.

To any one interested in the mathematical instruction in the United States, the following questions come at once to mind, after reading the Cairns report (see above): "What is the course of training which enables students to meet such tests?" and "Can any one of these methods be adopted to our system?"

An English inspector who visited schools in New York State and in Indiana, once remarked that "American education is like a broad highway with lines for many kinds of traffic, but no arrangement for rapid transit. Prof. Watkeys notes: "What we need is a 'third lane' which will permit the abler students to proceed at their own and not at the pace of the average. . . . More than anything else we need to offer our abler students as good an opportuuity relatively for them as we now offer our average students. That is, we need quality in the educational opportunity not identical opportunity. Only in this way can we be truly democratic."

Evidently all roads lead to homogeneous grouping.

9. Webber, W. Paul. *Individual study of mathematics.* National Mathematics Magazine. 9: 106–08. January 1935.

A discussion of various plans to encourage individual study of mathematics. "It appears that one of the first duties of the mathematics teacher in college [and of all teachers, in every grade, in every subject,—N.L.] is to teach the need of, and how to study from books. This is an individual matter

and is not so simple either. For there are many pleasant distractions on most campuses."

10. Wright, Harvey A. *Mathematics in the liberal arts college*. National Mathematics Magazine. 9: 95–99. January 1935.

An interesting exposition and a clear analysis of the arguments in favor of the position that "Mathematics has a place in the college curriculum other than for those students who expect to enter the engineering profession or for students who naturally like to study the subject."

Cogent as the given arguments appear, they suffer from a serious defect. They merely show that mathematics is *sufficient* for the inculcation of certain desirable modes of thought, but they do not prove that it is *necessary* i.e. *indispensable* for those objectives. Unless the latter is proved, the arguments of the natural scientists and even those of the social scientists go unanswered. We must therefore impress upon educators in general and upon the exponents of the social sciences in particular, that mathematics and logic are the *only* disciplines that exhibit the *general pattern of thinking* of which the propositions in the individual sciences are but particular exemplifications.

In the same manner as the processes of arithmetic become clearer to one who has studied algebra, so do the methods of reasoning in the sciences become more meaningful and thereby more powerful to one who has mastered some mathematics and logic.

◆ NEW BOOKS ◆

A Manual of Thesis Writing. (For Graduates and Undergraduates). By Arthur H. Cole and Karl W. Bigelow. John Wiley and Sons, 1934, IX+48. Price, $0.75.

It is the aim of the authors of this manual to offer students certain comments on the aims of scientific work and a convenient compilation of rules and usages which are widely accepted among mature writers as helpful in the preparation of first-class scientific papers.

Judging from the appearance of many of the manuscripts that the average student is willing to pass in for professorial criticism this book ought to serve a useful purpose if it can be put into the hands of such students. Not only can such students save a large amount of time by following such suggestions as are set forth in this manual, but the general quality of their theses will be improved.

Unit Mastery Mathematics. By John C. Stone, C. N. Mills and Virgil S. Mallory, Benj. H. Sanborn and Company, 1933, Book One, V+314. Price, $0.96. Book Two, 1934, V+432. Price $1.00. Book III, 1934; V+469. Price $1.20.

This new series of books will be of interest to all teachers who are already familiar with the previous texts prepared by the above authors. While one may not agree in every detail with their pre-

sentation one can always find many fine things of value in any of the books which they write. This is due in large part to the fact that they are teachers with a rich classroom experience.

There guiding principles were used in the preparation of the books:

1. Easy to learn.
2. Easy to teach.
3. Interesting to pupils.

The books conform in spirit to the most modern requirements as to what the junior high school course in mathematics should concern itself such as informal geometry, arithmetic, numerical trigonometry, and an approach to the more formal study of demonstrative geometry.

The books are well made, are most attractive in appearance and doubtless will be used in a large number of schools where no longer the old traditional course in arithmetic holds sway.

Introduction to Practical Astronomy. By Dinsmore Seter. Thomas Y. Crowell Company, 1933, VIII+80. Price, $2.00.

This book has been prepared so as to serve as a beginning course in any college with the most meager equipment. The book is based upon twenty years' experience in teaching astronomy which has led the author to give much of the work traditionally given by the lecture method in the laboratory. Consequently, all instruction in diurnal motions, coordinates, time, and the like is confined by the author to simple and inexpensive equipment adequate for a first course. All experiments requiring special astronomical apparatus are grouped in the back of the book. This makes the book equally usable to the teacher in the small college and in the university.

In general form the book is a laboratory manual although it also serves as a text.

The Teaching of Arithmetic. By Paul Klapper. D. Appleton-Century Company, 1934, XII+525. Price, $2.60.

This is the second edition of a book first published in 1916, but it is really more than a revised edition because the author has completely reconsidered arithmetic and its teaching from the standpoint of modern ideas and practice. Thus, he has had the benefit of all the advances that have been made during the last 19 years in improving educational practice.

The book presents a critical analysis of the social and educational values that have been claimed for arithmetic, gives a detailed discussion of curriculum problems, goes into the basic psychological factors that determine the most efficient methods of teaching, gives some thought to the problems of supervision, studies and evaluates critically the various lesson types and presents special teaching techniques in which the author doubtless believes.

The book was written primarily to be used as a text in courses in general methods and in special methods but this should not impair its value for the ordinary class room teacher or supervisor in the field.

Plane Geometry. By Joseph P. McCormack. D. Appleton-Century Company, 1934, XIV+455.

This book is a revision of the author's earlier edition. The exercises are intended to be comprehensive enough to meet varying needs and individual differences in ability have been considered in classifying the exercises.

New diagrams, halftones, units of solid geometry and other features have been added to improve the teaching pro-

cedure. Careful attention is given to functional relationships and a thorough treatment of numerical trigonometry is given. The book is made rather large by giving complete proofs to all propositions and it is doubtful whether there is a gain here because the pupil is expected to look up many of the reasons.

New type tests are included which ought to make the book more attractive to pupils. In the hands of a good teacher the book can be made a very efficient help.

Humanized Geometry. (An Introduction to Thinking.) By J. Herbert Blackhurst. Published by the author at Drake University, Des Moines, Ia., 1934, 206 pages. Price, $1.00 net, plus carrying charges.

As its name implies, this book is intended to humanize the study of geommetry by adjusting the contents to the needs and interests of the pupils. It is to serve as an introduction to thinking. The claim is often made that the study of geometry enables a boy or girl to think clearly, but does it? If this book can be used so as to make the pupil conscious of the type of thinking he is doing so that he will ultimately come to have ideals of procedure clearly developed, it will serve a very useful purpose.

The book is attractively bound and will be of interest to teachers who are anxious to see how such work is carried on in actual practice.

Tables of Integrals and other Mathematical Data. By H. B. Dwight. The Macmillan Company, 1934, VIII +222. Price, $1.50.

Teachers of mathematics will be interested to see this new book. It is not intended to be used as a first approach. The references, although not complete, are intended to help the reader to find out where the derivation of the results are given or where further similar results may be found.

Review of Pre-College Mathematics. By C. J. Lapp, F. B. Knight, and H. L. Rietz. Scott, Forsman and Company, 1934, IV+124.

This book is intended to provide explanatory work for and drill in those fundamental aspects of arithmetic, algebra, geometry, and trigonometry necessary for freshman college work in mathematics and science.

The contents of the book was worked out by actual experiment in the classroom and a subsequent research on ferquency of errors made by students in the freshman college classes.

The authors of the book are well known and represent the fields of mathematics, physics, and educational psychology. This should guarantee some kind of adequate correlation of the material.

It is of interest to see that teachers in the various fields of knowledge are beginning to try to correlate the work of their various fields in an intelligent manner.

The Poetry of Mathematics and Other Essays. By David Eugene Smith. Scripta Mathematica, Yeshiva College, New York, N.Y., 1934, IV+96. Price, 50¢ in heavy grey paper cover or 75¢ in handsome blue cloth binding.

This is the first book of a series of small volumes to be known as *The Scripta Mathematica Library.* The series will deal with the history and philosophy of mathematics, and with its relations with the other great activities oj the human spirit. The volumes are designed to furnish material which will interest not only students and teachers of mathematics, but also all who would like to resume the contact they had with the subject in their school or college days.

The articles in this first volume are not themselves mathematical but they relate to lines of interest which mathematics suggest. In these days when we hear so much about correlating the work of the various fields, the first article of this little book emphasizes the relationship between fine arts to mathematics through the vehicle of poetry in a fascinating way.

The second article on the *Call of Mathematics* ought to help many to at least hear the call of mathematics on the lower levels at least.

The third article on *Religio Mathematica* while not intended to develop mathematicians or to clarify religious dogma should lead thoughtful students to some serious thinking along interesting paths.

The fourth and fifth articles show the statesman Jefferson as deeply interested in mathematics, and the mathematician Monge as deeply interested in the affairs of state.

Teachers and friends of mathematics will find this volume a fine inspiration for developing in youth an abiding love of mathematics.

W. D. Reeve

College Algebra. By Walter B. Ford. The Macmillan Company, 1935, VII+304. Price, $1.90.

The author prepared this book with the following objectives in mind: (1) to bring college algebra into the closest possible contact with the affairs of daily life and, (2) to correlate the subject with those central facts from elementary geometry which the student must know at all times if he is to succeed in college mathematics.

The book is written with the various chapters independent of each other so as to adjust the book to a short or long course. Some of the more advanced and difficult topics such as partial fractions and limits have been omitted and an extended treatment of the subject of variation has been included.

Teachers of college algebra will want to see this new and interesting book.

Solid Geometry. By Frank M. Morgan and W. E. Breckenridge. Houghton Mifflin Company, 1934, VI+286. Price, $1.24.

This book follows the plan of the authors' Plane Geometry in its main features: (1) A wealth of original exercises, (2) Provision for individual differences, (3) A classified index of exercises, (4) Analysis of each proof before the formal demonstration, (5) Applied problems include practical ones from real life, and (6) modern tests.

The authors of the book are both successful classroom teachers which ought to insure that they know what type of course we should now present to pupils who want to study solid geometry.

It is somewhat strange that living in a world of three dimensions we teach for the most part a geometry of flatland. This book will help to remedy that defect if properly used.

Mechanics of Engineering. By S. D. Chambers. The Macmillan Company, 1934, VII+279. Price, $3.50.

This is another volume of the Engineering Science Series edited by Dr. E. R. Hedrick. This book is intended to be a text on the essentials of Statics and Kinetics and has been prepared definitely for students of Engineering who have had the necessary prerequisites in Calculus and College Physics.

The method of presentation has been based on the close correlation of three

teaching objectives: (1) the statement of a principle; (2) the proof or illustration of the principle, and (3) the application of the principle to illustrated examples and to related problems following the illustration.

The Arithmetic of Business. By Frank J. McMackin, John A. Marsh and Chas. E. Baten. Ginn and Company 1934, IX+486. Price, $1.48.

This new book is not only timely but it is attractive in appearance. If pupils are to be taught the essentials of business arithimetic, we must decide upon what these essentials are. This these authors claim to have done thoroughly. They have organized the material along psychological and pedagogical lines, they have provided an adequate testing program, including material for remedical instruction and they have provided for the maintenance of skills.

The book is the result of classroom experience and should be of interest to all teachers of arithimetic.

Government and Business. By Ford P. Hall. McGraw-Hill Book Company, 1934, X+275.

This book is an abbreviated form of what the author doubtless intends to enlarge and publish in permanent form at a later date. It is an attempt to survey the business field with regard to the increase in government control and aid to business. The book also deals with new legislation, with past federal regulation, and with state and municipal control.

According to the author's own statement "In the study of government, too much emphasis has been placed upon the rights of the individual and the organization of government, and too little attention has been given to the service which government performs."

A careful reading of the book should help to clear up some of these matters

Engineering Surveys. By Harry Richey. The Macmillan Company, 1934, XV+321. Price, $3.00.

This book is one of a series on Engineering Science edited by Dr. E. R. Hedrick and is the result of thirty years of engineering practice and teaching. It attempts to give a careful and adequate treatment of the subject from the standpoint of the modern engineer and engineer executive. It explains the fundamentals of engineering surveys, the procurement and use of maps and data, the organization and costs of surveys, and the like.

The text is planned to be equally useful to those who take further courses in surveying and those who will have no further instruction beyond the contents of this book.

Elements of The Differential and Integral Calculus. By W. A. Granville, Percy F. Smith, and W. R. Longley. Ginn and Company, 1934, XI+516.

Little need be said about a book which has preserved the main features which have made it so popular in the past. It is only natural as time goes on and new ideas come along books are improved by revision. New topics are introduced and some of the old ones no longer considered so important are omitted.

The book makes provisions for individual differences and furnishes extra problems for the more superior students at the end of most chapters.

A First Course in Educational Statistics. By Edna E. Kramer. John Wiley and Sons, Inc., 1935, IX+212. Price, $2.50.

This book is the result of an experimental course in educational statistics

with a heterogeneous group of pupils whose abilities, experiences, and interests were so varied that the author found existing texts unsatisfactory for her needs. This text, therefore, grew out of mimeographed notes and a lithoprinted text.

The author has carefully selected her illustrative material from current educational periodicals rather than to resort to artificial data. The book should be of interest to all teachers of educational statistics who want a book that has actually proved its worth in an actual classroom situation.

General Mathematics. By C. H. Currier and E. E. Watson. The Macmillan Company, 1931, VIII+413. Price, $3.50.

Until a short time ago general mathematics in the colleges did not make much headway. The reason for this is not altogether clear for a book of the character of this one ought to appeal more to freshmen college students than the traditional program of teaching the subjects separately.

For some high school graduates a part of the text will be in the nature of review but the review is not overdone.

The authors have chosen their exercise material with an eye to utility rather than to mathematical perfection. This is as it should be in a first course of this kind.

The use of books of this type should help to crystallize the best type of material for a first course in colleges and universities.

Progressive Plane Geometry. By Webster Wells and W. W. Hart. D. C. Heath and Company, 1935, IX+390. Price, $1.36.

The general appearance of this book is a distinct asset. If the soul of a book is as important as some people believe, then more attention must be given to this feature in the manufacture of texts.

As the name implies this book is intended to be modern and is designed to take care of individual needs. The long experience of the author as a classroom teacher of geometry ought to guarantee that the book is an improvement upon his earlier efforts. His attempt to teach geometry as a method of thinking is to be commended.

The introduction of solid geometry concepts along with those of plane geometry is a tendency in the right direction. Tests of a mastery type and also additional ones are given to be used by the teacher as he sees fit.

Differential Equations. By N. B. Conkwright. The Macmillan Company, 1934, XII+234. Price, $1.90.

This text is based upon a course given by the author for a number of years at the University of Iowa. It presupposes a working knowledge of the calculus. The author has made certain omissions calculated to make the text of greater interest and value to the student who uses it, but in such cases he has given reference to standard treatises where some of the omitted material can be found for those who wish to have it.

Most of the exercises in the text are new and have the added value of being arranged in psychological order.

In these days when educators are beginning to realize that knowledge of subject matter is the main support for a teacher this new book should find a place.

Field Work in Mathematics. By Carl N. Shuster and Fred L. Bedford. American Book Company, 1935, VII+168. Price, $1.20 list.

This book is the outgrowth of several

years of classroom experiment and teaching at Teachers College, Columbia University during the academic year and in the summer session. The authors have cooperated continuously to make this book the most interesting and helpful text on "Field Work in Mathematics" that has yet appeared.

There is no reason why in teaching mathematics some of the more practical phases of the subject should not be taught especially where such material throws light on the meaning of mathematics.

This book brings such ideas out in a most interesting way and should be in every mathematics classroom in the secondary schools of the country at least as a reference volume if not as a text.

Solid Geometry. By Elizabeth B. Cowley. Silver, Burdett and Company, 1934, VIII+230. Price, $1.28.

As long as solid geometry is to be taught as a separate subject in the schools books like this one should serve the purpose admirably. The book is illustrated better than traditional texts so as to bring out the idea that the geometry therein relates to a world of three dimensions and not entirely to flatland as most of our geometry courses fail to bring out.

Whether solid geometry continues to be taught as a separate course or not the fact remains that the teaching of the subject is rapidly passing out. If solid geometry or any part of it is worth saving it must be combined and taught with plane geometry or be improved as the author here has done by making it relate more and more to the practical affairs of life.

Differential and Integral Calculus. By J. H. Neelley and J. I. Tracey. The Macmillan Company, 1932, VIII+496. Price, $4.00.

This book is planned so as to meet the needs of both academic colleges and engineering schools. As a result any student who uses the book will get a fair appreciation of the wide applications of the calculus in modern science and in engineering, if he understands the underlying principles of the subject. The teacher in any case can decide upon the degree of emphasis to be placed upon theory and applications.

It is unfortunate that books of this type are so little used that the cost of them is made almost prohibitive for many students and teachers who might otherwise be interested to study the contents. Better books at lower prices will come only when the demand has been made strong enough.

A Critical Study of the Teaching of Elementary College Mathematics. By Joseph Seidlin. Bureau of Publications, Teachers College, Columbia University, New York, N. Y. Contributions to Education No. 286, 1931, VIII+107. Price $1.75 list.

The reviewer must confess that he belongs to that not yet quite extinct type of those who learned to teach by teaching without the advantage of pedagogical guidance or instruction in methods. He had enjoyed good teaching and endured other kinds with a measure of resignation, and just naturally sought to profit in practice from both kinds of experience. Now, a half-century later, there are better ways—and perhaps worse.

In serving on various committees, the reviewer, like his associates, has indulged in more or less edifying reflections on how teaching should be done and how it should be organized, yet too often with at least a subconsciousness that the

generalities were of dubious glitter, that their luminosity was rather that of the elusive aurora than let us say of the lowlier "brass tack."

Dr. Seidlin's book is a welcome contribution to the concrete need of an objective critical discussion of actual teaching of a specific subject. A logical educational theory underlies his opening classification, which may be briefly summarized as follows:

(1) Lecture, discursive and entertaining; students passively interested; (2) Lecture, more concentrated, with active questioning attention; (3) Give and take discussion; (4) Give and take discussion with blackboard work by students; (5) Text-book-repeating recitation; (6) Planned question and answer development; (7) Blackboard recitation, graded.

It seems highly significant that of 150 cases 61 are of type 7, 45 of type 5—these seeming to require the least mental effort from the teacher. They are justified in that "they bring results"—a dictum with which the reviewer certainly agrees, with his own interpretation of "results," however.

The following hundred pages, after copious and apt illustration, deal judiciously and concisely with the general topics, Violation of Some Principles of the Learning Process; What Is Good Teaching? The Examination and the Textbook; The Use of the Question; Productive Research and Teaching; An Evaluation of Attempts to Improve the Teaching of Elementary College Mathematics; Summary and Conclusions. While Dr. Seidlin confesses to a strong prejudice in favor of research he cites significant instances of serious errors in teaching on the part of men so absorbed in research that they became negligent of essential elementary facts.

One point not mentioned which the reviewer ventures to believe of some importance is the preliminary discussion of the lesson assigned for the following exercise, with a view not of relieving the student of desirable home study but of assisting him in making it effective. The adaptation of any method to the mixed capacities of ungraded sections is perhaps outside the proper scope of the author's plan. The dependence of the choice of method on the particular subject matter is also another story.

The author concludes with the fitting remark: "Future research in the teaching of elementary college mathematics may broaden or help to establish the findings of this study, which have been submitted with the customary hopes, shortcomings and limitations of pioneer efforts."

The reviewer would recommend a careful perusal of this little book to every teacher who has not acquired perfection in the art of teaching, or the rigidity which precludes further progress. May similar manuals appear in other fields.

H. W. Tyler
